New classes of potentials for which the radial Schrödinger equation can be solved at zero energy: II

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Abstract

In a previous paper (Chadan and Kobayashi 2006 *J. Phys. A: Math. Gen.* **39** 3381), we presented an infinite class of potentials for which the radial Schrödinger equation at zero energy can be solved explicitly. For part of them, the angular momentum must be zero, but for the other part (also infinite), one can have any angular momentum. In the present paper, we study first a simple subclass (also infinite) of the whole class for which the solution of the Schrödinger equation is simpler than in the general case. This subclass is obtained by combining another approach together with the general approach of the previous paper. Once this is achieved, one can then see that one can in fact combine the two approaches in full generality, and obtain a much larger class of potentials than the class found in Chadan and Kobayashi (2006). We mention here that our results are explicit, and when exhibited, one can check in a straightforward manner their validity. The main result of the present paper is given in theorem 2.

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1. Introduction

In a recent paper [1], we showed how, starting from two regular potentials for which the radial Schrödinger equation can be solved explicitly at zero energy—there are many of them [1, 2]—one can construct explicitly an infinite number of potentials for which one can solve again, explicitly, the radial equation at zero energy. The solutions for these new potentials are given very simply in terms of the solutions of the two initial potentials. By construction, it is then seen that the method can be applied without any modifications to potentials which are

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singular (repulsive) at the origin, or are long range (Coulomb, etc), or are even confining, such as λr^2, $\lambda > 0$, etc. The main result of the above paper is shown in theorem 1 below. According to the case, one can include, as well, the angular-momentum potential $\ell(\ell + 1)/r^2$. Many examples, both for regular and singular potentials, covering all cases, except the confining potentials, are given in [1], with explicit solutions of the radial equation.

In the present paper, following a different method, we study first a subclass of the general class, which is simpler, and more explicit. As in the general case, one can have, here too, singular (repulsive) potentials at the origin, or long-range potentials such as Coulomb potential, etc. We then generalize theorem 1 to the case where one has three potentials at the start, and show that the radial Schrödinger equation at zero energy can be solved explicitly for a much larger class of potentials than shown in the above theorem. Our new results are summarized in theorem 2, after formula (52).

For the convenience of the reader, we give here a résumé of [1], where all the proofs can be found, together with appropriate references. One can find there many explicit examples.

Consider first the radial Schrödinger equation at zero energy for the S-wave

$$\begin{cases} \varphi_0''(r) = V_0(r)\varphi_0(r), \\ r \in [0, \infty), \quad \varphi_0(0) = 0, \quad \varphi_0'(0) = 1. \end{cases} \tag{1}$$

It is assumed that either V_0 is positive (repulsive), or else, if it is negative (attractive), it is weak and sustains no bound states. Moreover, we assume

$$\int_0^1 r|V_0(r)| \, dr < \infty, \qquad \int_1^\infty r^2|V_0(r)| \, dr < \infty. \tag{2}$$

Under the above conditions on $V_0(r)$, i.e. absence of bound states and (2), it can be shown that [1–3]

$$\begin{cases} \varphi_0(r) = r + o(r) & \text{as} \quad r \to 0, \quad \varphi_0(r) > 0 \quad \text{for all} \quad r > 0, \\ \varphi_0(r) = Ar + B + o(1) & \text{as} \quad r \to \infty, \quad 0 < A < \infty, \quad |B| < \infty. \end{cases} \tag{3}$$

A second independent solution of the Schrödinger equation is given by

$$\begin{cases} \chi_0(r) = \varphi_0(r) \int_r^\infty \dfrac{dt}{\varphi_0^2(t)}, & r > 0, \\ \chi_0(0) = 1, & W[\varphi_0, \chi_0] = \varphi_0'\chi_0 - \chi_0'\varphi_0 = 1. \end{cases} \tag{4}$$

Indeed, it follows from the definition of $\chi_0(r)$ that $\chi_0'' = V_0\chi_0$, and $W[\varphi_0, \chi_0] = 1$. Then $\chi_0(0) = 1$ follows from the first line of (3). One gets then

$$\begin{cases} \chi_0(r) > 0 & \text{for} \quad r \in [0, \infty), \\ \chi_0(r) = \dfrac{1}{A} + o(1) & \text{as} \quad r \to \infty, \end{cases} \tag{5}$$

where A, strictly positive and finite, is defined in (3).

Now, consider the mapping $r \to x(r)$ defined by

$$x(r) = \frac{\varphi_0(r)}{\chi_0(r)}, \qquad \frac{dx}{dr} = \frac{\varphi_0'\chi_0 - \varphi_0\chi_0'}{\chi_0^2(r)} = \frac{1}{\chi_0^2(r)} > 0. \tag{6}$$

It is obvious that the mapping is one-to-one, and is smooth. It is in fact C^2 since [1]

$$\frac{d^2x}{dr^2} = \frac{-2\chi_0'(r)}{\chi_0^3(r)}. \tag{7}$$

Therefore, we can use $x(r)$ for making a change variable in the Schrödinger equation. Note that one has also, according to (3) and (5),

$$\begin{cases} x(r) = r + o(r) & \text{as} & r \to 0, \\ x(r) = A^2 r + AB + o(1) & \text{as} & r \to \infty. \end{cases} \tag{6'}$$

We consider now the equation

$$\begin{cases} \varphi''(r) = V_0(r)\varphi(r) + [\chi_0(r)]^{-4} V[x(r)]\varphi(r), \\ \varphi(0) = 0, \quad \varphi'(0) = 1. \end{cases} \tag{8}$$

Note here that, as we saw before, $\chi_0(r)$ is a smooth, bounded and strictly positive function for all $r \geqslant 0$.

We assume again that

$$\begin{cases} \text{(i)} & V_0(r) \text{ satisfies (2), and sustains no bound states,} \\ \text{(ii)} & V(x) \text{ satisfies also (2) in the variable } x, \\ \text{(iii)} & V(r) \text{ can have any (finite) number of bound states.} \end{cases} \tag{9}$$

If we make now in (8) the change of variable $r \to x(r)$ defined by (6), and the change of function

$$\psi(x) = \left[\frac{\varphi(r)}{\chi_0(r)} \right]_{r=r(x)}, \tag{10}$$

$r(x)$ being the inverse mapping, well defined and also C^2, mapping $x \in [0, \infty)$ into $r \in [0, \infty)$, we find

$$\begin{cases} \ddot{\psi}(x) = V(x)\psi(x) \\ \psi(0) = 0, \quad \dot{\psi}(0) = 1. \end{cases} \tag{11}$$

Therefore, if (1) and (11) can be solved explicitly for $V_0(r)$ and $V(x)$, then (8) also can be solved explicitly, and we have, according to (10),

$$\varphi(r) = \chi_0(r)\psi(x) = \chi_0(r)\psi\left(\frac{\varphi_0(r)}{\chi_0(r)}\right). \tag{12}$$

We have therefore, the following theorem [1]:

Theorem 1. *Suppose the Schrödinger equation $\varphi''(r) = v(r)\varphi(r), r \in [0, \infty)$, together with $\varphi(0) = 0$, $\varphi'(0) = 1$, can be solved explicitly for two potentials $V_0(r)$ and $V(r)$, both satisfying the integrability conditions shown in (2). We assume (i) V_0 sustains no bound states; (ii) V can have bound states, their number n being finite. Then (8) also can be solved explicitly, and its solution is given by (12), where $\psi(x)$ is the solution of (11). This, of course, can be checked directly by differentiating (12).*

Remark 1.
Iteration. Once we have (12), we can start now with V_0 and $V_0 + \chi^{-4}(r)V[x(r)]$, and repeat the operation to get an infinite number of potentials.

Bound states. We have assumed that V_0 has no bound states. If $V(x)$ sustains n bound states, then, according to the nodal theorem [4, 5], $\psi(x)$, solution of (11), has n zeros for $x > 0$. It follows then from (12) that $\varphi(r)$ has the same number of zeros for $r > 0$, and, therefore, that the potential $V_0(r) + \chi_0^{-4}(r)V(x)$ sustains the same number of bound states as $V(r)$.

Remark 2. All the above formulae can be generalized for higher angular momenta. We refer the reader to [1]. As is explained there, one must replace now the second part of (2) by $r^{2\ell+2}V_0(r) \in L^1(1, \infty)$. One finds there also singular potentials containing logarithmic terms with explicit solutions.

2. A restricted class of potentials

For this purpose, we use a transformation of the Schrödinger equation devised some years ago by one of the authors and Harald Grosse, which has very smoothing effects on the potential [6]. It is as follows. Consider the radial Schrödinger equation at zero energy

$$
\begin{cases}
\varphi''(r) = V_0(r)\varphi(r), & V_0 \text{ of any sign}, \\
r \in [0, \infty), & \varphi(0) = 0.
\end{cases}
\tag{13}
$$

$V_0(r)$ is assumed to be a regular potential, i.e. to satisfy the Bargmann–Jost–Kohn condition [2, 3]

$$
r V_0(r) \in L^1(0, \infty).
\tag{14}
$$

We introduce now

$$
W_0(r) = -\int_r^\infty V_0(t)\, \mathrm{d}t; \qquad U_0(r) = \int_r^\infty W_0(t)\, \mathrm{d}t.
\tag{15}
$$

Our new transformation is now defined by [6]

$$
x = \int_0^r \mathrm{e}^{2U_0(t)}\, \mathrm{d}t, \qquad \psi(x) = \mathrm{e}^{U_0(r)}\varphi(r).
\tag{16}
$$

Remark 3. Since, for large values of r, we have $|V_0| < r|V_0| \in L^1(\infty)$, W_0 is well defined and is an absolutely continuous function for all $r > 0$. Then it is easily seen that $W_0(r) \in L^1(0, \infty)$. Indeed,

$$
|U_0(r)| = \left|\int_r^\infty W_0(r)\, \mathrm{d}r\right| \leqslant \int_0^\infty |W_0(r)|\, \mathrm{d}r \leqslant \int_0^\infty \mathrm{d}r \int_r^\infty |V_0(t)|\, \mathrm{d}t
$$
$$
= \int_0^\infty |V_0(t)|\, \mathrm{d}t \int_0^t \mathrm{d}r = \int_0^\infty t\, |V_0(t)|\, \mathrm{d}t < \infty.
\tag{17}
$$

It follows that $U_0(r)$ is a well-defined, bounded and absolutely continuous function for all $r \geqslant 0$, and is also continuously differentiable for $r > 0$.

From the above results, it is obvious that the transformation (16) is a nice smooth transformation, and we have a smooth one-to-one mapping

$$
r \in [0, \infty) \quad \Leftrightarrow \quad x \in [0, \infty),
\tag{18}
$$

and $(\mathrm{d}x/\mathrm{d}r) = \exp(2U_0(r)) > 0$. Obviously, we have also

$$
\varphi(0) = 0 \quad \Leftrightarrow \quad \psi(0) = 0.
\tag{19}
$$

Making the transformation (16) in (13), we find, with $\dot{} = \frac{d}{\mathrm{d}x}$,

$$
\begin{cases}
\ddot{\psi}(x) = \left[-W_0^2(r)\, \mathrm{e}^{-4U_0(r)}\right]\psi(x) = \widetilde{V}(x)\psi(x) \\
x \in [0, \infty), \qquad \psi(0) = 0.
\end{cases}
\tag{20}
$$

Now, since $U_0(r)$ is a very smooth and bounded function, for all $r \geqslant 0$, and $U(\infty) = 0$, it is obvious from the definition (16) that x and r are very close to each other, and we have [6]

$$
\begin{cases}
x \simeq \mathrm{e}^{2U_0(0)}r + o(r), & \text{as } r \to 0, \\
x \simeq r + O(1), & \text{as } r \to \infty.
\end{cases}
\tag{21}
$$

Therefore, as far as integrability at $x = 0$ and $x = \infty$ are concerned, we have

$$
\int_0^\infty x\left[\mathrm{e}^{-4U_0(r)} W_0^2(r)\right] \mathrm{d}x \sim \int_0^\infty r W_0^2(r)\, \mathrm{d}r.
\tag{22}
$$

To show that the last integral is absolutely convergent is now very easy. Indeed, from the definition of $W_0(r)$, (15),

$$|r W_0(r)| \leqslant \int_r^\infty t \, |V_0(t)| \, \mathrm{d}t \, < \, \int_0^\infty t \, |V_0(t)| \, \mathrm{d}t = C < \infty. \tag{23}$$

Therefore,

$$\left| r W_0^2(r) \right| \leqslant C |W_0(r)|, \tag{24}$$

and since $W_0(r)$ was shown to be $L^1(0, \infty)$, the same is true for $r W_0^2(r)$. Therefore, in (20), $x \widetilde{V}(x) \in L^1(0, \infty)$, and $\widetilde{V}(x)$ is a regular potential.

Consider now the equation

$$\varphi''(r) = \left[V_0(r) + W_0^2(r) + \mathrm{e}^{4U_0(r)} V_1(x) \right] \varphi(r)$$
$$= V(r)\varphi(r). \tag{25}$$

Note here the variable $x = x(r)$ in $V_1(x)$! Both $r V_0(r)$ and $x V_1(x)$ are assumed to be $L^1(0, \infty)$, i.e. satisfy (14). Since we showed that $r W_0^2(r)$ was also $L^1(0, \infty)$, and x and r are always of the same order by virtue of (21), the full potential in (25) satisfies the same integrability condition.

After making the transformation (16), one finds then, with $\dot{} = (\mathrm{d}/\mathrm{d}x)$,

$$\begin{cases} \ddot{\psi}(x) = V_1(x)\psi(x), \\ x \in [0, \infty), \quad \psi(0) = 0, \quad x V_1(x) \in L^1(0, \infty). \end{cases} \tag{26}$$

Therefore, if equation (26) with $V_1(x)$ can be solved explicitly, we can solve also explicitly (25), and its solution is given by

$$\varphi(r) = \mathrm{e}^{-U_0(r)} \psi(x), \tag{27}$$

where x is defined in (16). Therefore, we have completed our programme, i.e. starting from two potentials V_0 and V_1, and knowing that the Schrödinger equation can be solved explicitly for V_1, to find a new potential for which the same holds. One can check again, directly, that (27) is, indeed, a solution of (25).

The connection with [1] is as follows. Consider

$$\psi_0'' = \left(V_0 + W_0^2 \right) \psi_0. \tag{28}$$

A solution of this equation, called $\chi_0(r)$, with $\chi_0(0) = 1$, is

$$\chi_0(r) = \frac{\mathrm{e}^{-U_0(r)}}{\mathrm{e}^{-U_0(0)}}. \tag{29}$$

This has no zeros for $r \geqslant 0$. Therefore, the physical solution of (28), called $\varphi_0(r)$, with $\varphi_0(0) = 0$, and given by [1]

$$\varphi_0(r) = \chi_0(r) \int_0^r \frac{\mathrm{d}t}{\chi^2(t)}, \tag{30}$$

as can be checked easily, has also no zeros for $r > 0$. Therefore, according to the nodal theorem [4, 5], $V_0 + W_0^2$ cannot have bound states, whatever the sign of V_0 is. According to (29), the full potential in (25) can be written as

$$V(r) = \left(V_0(r) + W_0^2(r) \right) + \lambda \chi_0^{-4}(r) V_1(x). \tag{31}$$

This is in complete analogy with [1], where we had

$$V(r) = V_0(r) + \lambda \chi_0^{-4}(r) V_1(x), \tag{32}$$

assuming there $V_0(r)$ to be either positive, or negative but weak enough in order not to have bound states in (13). In the present paper, V_0 is replaced by $V_0 + W_0^2$, with the same properties,

but now with simple explicit solutions given by (29) and (30), and no restrictions on the sign of V_0.

Bound states. So far, we have not assumed anything on the signs of V_0 and V_1 in (25). They can have any sign. As we showed, $V_0 + W_0^2$ alone cannot have bound states. So in order to have bound states, $V_1(x)$ must be negative, and strong enough. Now, again, the nodal theorem [4, 5], applied to (26), shows that, if $V_1(x)$ sustains n bound states, then $\psi(x)$ has n zeros (nodes) for $x > 0$. Therefore, according to (27), $\varphi(r)$ also has n nodes for $r > 0$. So, there are n bound states also for the full potential (31). In conclusion, whatever the sign of V_0 is, $(V_0 + W_0^2)$ sustains no bound states, and $V_0 + W_0^2 + \lambda \chi_0^{-4} V_1(x)$ and $V(r)$ have the same number of bound states.

Higher ℓ. Generalization to include the centrifugal potential $\ell(\ell + 1)/r^2$ in $V_0 + W_0^2$ is straightforward [6]. We define now

$$\begin{cases} W_\ell(r) = -\int_r^\infty V_0(t) t^{-2\ell} \, dt, \\ U_\ell(r) = \int_r^\infty W_\ell(t) t^{2\ell} \, dt. \end{cases} \tag{33}$$

Since

$$|r^{2\ell} W_\ell(r)| \leqslant \int_r^\infty V_0(t) \, dt < \infty, \qquad \forall r > 0, \tag{34}$$

it is obvious that $r^{2\ell} W_\ell(r)$, like $W_0(r)$, is bounded for $r > 0$, and $\in L^1(0, \infty)$. And $U_\ell(r)$ is a nice bounded and smooth function for all $r \geqslant 0$, as was the case for $U_0(r)$. Now, we define,

$$\chi_\ell(r) = r^{-\ell} e^{-U_\ell(r)}. \tag{35}$$

It is then easily seen that $\chi_\ell(r)$ satisfies

$$\chi_\ell''(r) = \frac{\ell(\ell + 1)}{r^2} \chi_\ell(r) + V_0 \chi_\ell + r^{4\ell} W_\ell^2(r) \chi_\ell(r). \tag{36}$$

The generalization of (25) is now

$$\varphi_\ell''(r) = \left[V_0(r) + r^{4\ell} W_\ell^2(r) + \frac{\ell(\ell + 1)}{r^2} + r^{4\ell} e^{4U_\ell} V_1(x) \right] \varphi_\ell(r), \tag{37}$$

and one finds, after the transformations

$$x = x(r) = \int_0^r t^{2\ell} e^{2U_\ell(t)} \, dt, \qquad \psi_\ell(x) = r^\ell e^{U_\ell(r)} \varphi_\ell(r), \tag{38}$$

the differential equation

$$\ddot{\psi}_\ell(x) = V_1(x) \psi_\ell(x). \tag{39}$$

Therefore, the conclusion is exactly as for the case $\ell = 0$. Remember that we have assumed $x V_1(x) \in L^1(0, \infty)$. The solution of (37) is, therefore, given by

$$\varphi_\ell(r) = r^{-\ell} e^{-U_\ell(r)} \psi_\ell(x). \tag{37'}$$

Making $\ell = 0$, we find, of course, (25)–(27).

3. The general case

In this section, we are going to combine both transformations, namely, the transformation of section 1, and the transformation of section 2. Consider the three potentials

$$
\begin{cases}
V_0(r) \text{ satisfying (2): } r V_0(r) \in L^1(0, 1), \quad r^2 V_0(r) \in L^1(1, \infty), \\
\quad \text{and no bound states;} \\
V_1(r) \text{ satisfying } r V_1(r) \in L^1(0, \infty); \\
V(r) \text{ satisfying } r V(r) \in L^1(0, \infty). \\
V_1(r) \text{ and } V(r) \text{ can have bound states.}
\end{cases}
\tag{40}
$$

We assume that one can solve explicitly

$$
\begin{cases}
\varphi''(r) = v(r)\varphi(r), \qquad v = V_0(r), V(r), \\
\varphi(0) = 0, \qquad\qquad \varphi'(0) = 1.
\end{cases}
\tag{41}
$$

Nothing is assumed for the explicit solution for $v = V_1(r)$.

As we saw in section 1, the assumptions on $V_0(r)$, entail the existence of the solution $\chi_0(r)$ of (41), given by (4), such that

$$
\begin{cases}
\chi_0(r) > 0 \quad \text{for all} \quad r \geqslant 0, \\
\chi_0(0) = 1, \quad \chi_0(\infty) = \dfrac{1}{A}, \quad 0 < A < \infty,
\end{cases}
\tag{42}
$$

that is, a smooth and strictly positive bounded function. We introduce now

$$
\begin{cases}
W_1(r) = -\displaystyle\int_r^\infty V_1(t)\chi_0^2(t)\, dt, \\
U_1(r) = \displaystyle\int_r^\infty W_1(t)\dfrac{dt}{\chi_0^2(t)}.
\end{cases}
\tag{43}
$$

It now follows that W_1 and U_1 have the same properties as W_0 and U_0 introduced earlier in (15), and shown in remark 3, (17) and (24):

$$
\begin{cases}
W_1(r) \in L^1(0, \infty), \\
U_1(r) \text{ smooth and bounded for all } r \geqslant 0, \\
r W_1^2(r) \in L^1(0, \infty).
\end{cases}
\tag{44}
$$

Let us now introduce, as in section 2, the mapping

$$
r \to x = x(r) = \int_0^r e^{2U_1(t)}\frac{dt}{\chi_0^2(t)}.
\tag{45}
$$

It is a smooth, and twice differentiable one-to-one mapping for $r > 0$:

$$
\begin{cases}
r \in [0, \infty) \Leftrightarrow x \in [0, \infty), \\
r = 0 \Leftrightarrow x = 0, \quad r = \infty \Leftrightarrow x = \infty.
\end{cases}
\tag{46}
$$

Indeed, $U_1(\infty) = 0$, $\chi_1(\infty) = \frac{1}{A} \neq \infty$, as shown in (5), and

$$
\begin{cases}
\dfrac{dx}{dr} = \dfrac{e^{2U_1(r)}}{\chi_0^2(r)} > 0, \qquad \to A^2 \quad \text{for} \quad r \to \infty, \\
\dfrac{d^2x}{dr^2} = -\left[\dfrac{2W_1(r)}{\chi_0^4(r)} + \dfrac{2\chi_0'(r)}{\chi_0^3(r)}\right] e^{2U_1(r)} \text{ bounded for all } r > 0.
\end{cases}
\tag{47}
$$

Remember that $\chi_0'' = V_0\chi_0$. χ_0 being continuous and bounded for all $r \geqslant 0$, and $V_0 \in L^1$ by assumption for all $r > 0$, the same is true for χ_0'': $\chi_0''(r) \in L^1$ for $r > 0$. Therefore, $\chi_0'(r)$ is bounded and continuous for $r > 0$. The inverse mapping $r(x)$ is also, of course, a smooth and twice differentiable mapping for $x > 0$. Also, it is obvious from the first part of (47), that

$$
\begin{cases}
x(r) = e^{2U_1(0)}r + o(1), & r \to 0, \\
x(r) = A^2 r + o(r), & r \to \infty.
\end{cases}
\tag{48}
$$

x and r are, therefore, of the same order as $r \to 0$ or $r \to \infty$.

Consider now the Schrödinger equation at zero energy

$$
\begin{cases}
\varphi''(r) = \left[V_0(r) + V_1(r) + \chi_0^{-4}(r) W_1^2(r) + \chi_0^{-4}(r) e^{4U_1(r)} V(x) \right] \varphi(r), \\
\varphi(0) = 0.
\end{cases}
\tag{49}
$$

Note here, again, the appearance of x, given by (45), in $V(x)$. Remember also that $r W_1^2(r) \in L^1(0, \infty)$, as shown in (44). It follows that all the potentials in (49) are regular, i.e. $rv(r) \in L^1(0, \infty)$ by assumption. Making the change of function, where U_1 is defined by (43),

$$
\begin{cases}
\varphi(r) \to \psi(x) = \left[\dfrac{\varphi(r) e^{U_1(r)}}{\chi_0(r)} \right]_{r=r(x)}, \\
\varphi(0) = 0 \Leftrightarrow \psi(0) = 0, \text{ which is obvious,}
\end{cases}
\tag{50}
$$

differentiating twice $\psi(x)$ with respect to x defined by (45), and using the first part of (47) and (49), we find

$$
\begin{cases}
\ddot{\psi}(x) = V(x)\psi(x). \\
\psi(0) = 0.
\end{cases}
\tag{51}
$$

Since it was assumed that this equation can be solved explicitly, we have achieved our goal, and the solution of (49), according to (50), is given by

$$
\varphi(r) = \chi_0(r) e^{-U_1(r)} \psi(x),
\tag{52}
$$

where x is explicitly defined by (45) in terms of $\chi_0(r)$ and $V_1(r)$ through (43). We can summarize our results in the following theorem:

Theorem 2. *Given three potentials V_0, V_1, and V satisfying the assumptions (40), and assuming that (41) can be solved explicitly for V_0 and V, the solution of (49) is given explicitly by (52). Obviously, we have $\varphi(0) = 0$.*

Final remarks. Making $V_0(r) = 0$, i.e. $\chi_0(r) = 1$, we find the results of section 2, and making $V_1(r) = 0$, the results of [1], as given in the introduction. Also, having now a new potential with the explicit solution φ, we can repeat the operation with V_0 and the new potential, and continue indefinitely the process. We should note here that all the explicit examples of potentials we have given in [1] lead to the soluble Schrödinger equation for any coupling constant in front of the potential. We have, therefore, a great (infinite) variety of soluble potentials. Also, as we said in the abstract, once the solutions are exhibited, one can check directly, by differentiation, that they satisfy indeed the appropriate equations. One can, of course, include also here the angular momentum, either by proceeding as in section 2, or else by making

$$
\begin{cases}
V_0(r) \to V_0(r) + \dfrac{\ell(\ell+1)}{r^2}, & \ell \geqslant 0. \\
rV_0 \in L^1(0, 1), & r^{2\ell+2} V_0(r) \in L^1(1, \infty)
\end{cases}
\tag{53}
$$

in (41), and replacing $\varphi_0(r)$ and $\chi_0(r)$ by the appropriate solutions of $\varphi_\ell(r)$ and $\chi_\ell(r)$. Details are given in [1]. Since V_0 is assumed to sustain no bound states, the same is true when one makes (53). And one shows again that the mapping $r \rightarrow x(r)$ is one to one, and twice differentiable. One has $r \in [0, \infty) \Leftrightarrow x \in [0, \infty)$, and one can proceed as shown before in this section by replacing $\chi_0(r)$ by $\chi_\ell(r)$ in (43), (45), (49) and (50). We obtain now another kind of potentials with explicit solutions.

As one can imagine, many attempts have been made in the past to find potentials for which the Schrödinger equation can be solved explicitly, sometimes with specific properties of the solutions. We mention here the more recent papers, where full references can be found to earlier works [7–10]. Concerning our present work, the interest is mainly in the smoothing properties of the transformations of the Schrödinger equation we present in sections 1 and 2. These nonlinear transformations transform indeed, the Schrödinger equation with potentials which are supposed to be only locally L^1, to a similar equation with a smooth potential (equation (20)), or with smooth factor $\chi_0^4(r)$ in front of other potentials (equations (8) and (11)). This property can be used for finding easily low-energy expansions or high-energy asymptotic forms of phase shifts, etc. A paper on these problems is now under preparation.

Acknowledgments

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